JOURNAL OF APPROXIMATION THEORY 39, 1–10 (1983)

On Power Series with Gaps

P. SZÜSZ

Department of Mathematics, State University of New York, Stony Brook, New York 11794, U.S.A., and Mathematisches Institut A der Universität Stuttgart, Stuttgart, West Germany

Communicated by Oved Shisha

Received August 11, 1980; revised November 24, 1982

A theorem of Ingham refining one of Wiener [8] states that if

$$f(z) = a_0 + a_1 z + \cdots \tag{1}$$

is a power series with radius of convergence 1,

$$a_k = 0 \qquad \text{if} \quad k \neq k_n, \tag{2}$$

$k_1 < k_2 < \cdots$, being a sequence satisfying the inequality

$$k_{n+1} - k_n \geqslant A > 1 \tag{3}$$

and $f(e^{i\theta}) = \lim_{r \to 1^+} f(re^{i\theta})$ exists a.e., then there is a positive constant $K = K(A)$ such that

$$\infty > \int_I |f(e^{it})|^2 \, dt \geqslant K(A) \int_0^{2\pi} |f(e^{it})|^2 \, dt \tag{4}$$

for every interval I of length $\geqslant 2\pi/A$.

One could ask whether, in (4), the power 2 can be replaced by q's $\neq 2$. This problem was solved for $q > 2$ in the negative in 1962 by Erdös and Rényi [1] (cf. Zygmund [9, Vol. I, p. 380]).

Erdös and Rényi proved that for any $q > 2$ there is a function $f(z)$ such that even a condition stronger than (3), namely,

$$k_{n+1} - k_n \to \infty \qquad (n \to \infty) \tag{3'}$$

is satisfied; furthermore,

$$f(e^{2\pi it}) = O(1), \qquad \delta < t < 1 - \delta \quad (0 < \delta < \tfrac{1}{2}); \tag{5}$$

0021-9045/83 $3.00

but

$$f(e^{2\pi it}) \notin L_q(0, 1). \tag{6}$$

The proof of this theorem is based on probability theory and gives no example of a function having the properties (2), (3′), (5), and (6). An example of such a function was essentially given by Turán [7]. However, his construction works only for $q > 6$; further, property (5) is replaced by the weaker property

$$f(e^{2\pi it}) \in L_q(\tfrac{1}{4}, \tfrac{3}{4}). \tag{5'}$$

Later Knapowski [4] modified Turán's construction and replaced $q > 6$ by $q > 3$.

In Section 2 of the present paper I construct a function satisfying (2), (3′), (5), and (6) for any given $q > 2$.

1. Lemmas

Lemma 1. *Let t*

$$0 < t < 1, \tag{1.1}$$

let k_1 and k_2 be natural numbers satisfying

$$\frac{1}{k_1} < t < 1 - \frac{1}{k_1}, \qquad \frac{1}{k_2} < t < 1 - \frac{1}{k_2}, \tag{1.2}$$

$$(k_1, k_2) = 1, \tag{1.3}$$

and let

$$\|k_1 t\| < \frac{1}{10k_1}, \qquad \|k_2 t\| < \frac{1}{10k_2}, \tag{1.4}$$

where $\|y\|$ denotes the distance of the real number y from a nearest integer.

Then if $k_2 > k_1$, we have

$$\frac{k_2}{k_1} > 5 + \sqrt{24} \tag{1.5}$$

so that (1.3) and (1.4) cannot happen simultaneously too frequently.

Proof. Equations (1.2)–(1.4) yield, with some integers d_1 and d_2,

$$\frac{1}{k_1 k_2} \leqslant \frac{d_1}{k_1} - \frac{d_2}{k_2} \leqslant \left|\frac{d_1}{k_1} - t\right| + \left|t - \frac{d_2}{k_2}\right| \\ < \frac{1}{10}\left(\frac{1}{k_1^2} + \frac{1}{k_2^2}\right). \tag{1.6}$$

Set

$$k_2 = ck_1.$$

Then (1.6) implies

$$\frac{1}{c} < \frac{1}{10}\left(1 + \frac{1}{c^2}\right),$$

i.e.,

$$c^2 - 10c + 1 > 0,$$

which proves Lemma 1.

The following lemmas are from the theory of continued fractions. Let t be an irrational number, $0 < t < 1$, and let $[0; a_1, \dots]$ be its continued fraction expansion. I use Perron's [6] notation $A_0 = 0$, $B_0 = 1$; for $n = 1, 2, \dots$,

$$\frac{A_n}{B_n} = [0; a_1, \dots, a_n] \qquad (A_n > 0, B_n > 0, (A_n, B_n) = 1),$$
$$\zeta_n = [a_n; a_{n+1}, \dots];$$

and for $n = 0, 1, 2, \dots$,

$$D_n = B_n t - A_n = \frac{(-1)^n}{B_n \zeta_{n+1} + B_{n-1}} \qquad (B_{-1} = 0). \tag{1.7}$$

LEMMA 2 (Ostrowski [5]). *Any integer $m \geqslant 0$ has a unique representation*

$$m = \sum_{k=0}^{n} c_{k+1} B_k, \tag{1.8}$$

where the c_{k+1} are integers satisfying

$$0 \leqslant c_{k+1} \leqslant a_{k+1}$$

and

$$c_{k+1} = a_{k+1} \Rightarrow c_k = 0.$$

The proof is by induction.

LEMMA 3. *If $m \geqslant 1$ is an integer, then either*

$$\|mt\| > \|t\|$$

or

$$\|mt\| = \left| \sum_{k=0}^{n} c_{k+1} D_k \right|. \tag{1.9}$$

Proof. It suffices to show that either

$$\left\| \sum_{k=0}^{n} c_{k+1} D_k \right\| > \|t\|$$

or

$$\left| \sum_{k=0}^{n} c_{k+1} D_k \right| < \tfrac{1}{2}$$

which, in turn, is an easy consequence of (1.7) and the well-known formulae of the theory of continued fractions

$$D_{n+1} = a_{n+1} D_n + D_{n-1} \tag{1.10}$$

and

$$D_{n+1} = -\frac{1}{\zeta_{n+2}} D_n; \tag{1.11}$$

see Perron [6, pp. 4 and 36].

LEMMA 4. *Let N be a natural number, let ε be a given positive number, and let t be a given positive irrational number with the continued fraction expansion $[0; a_1 \cdots]$. Let l be the largest k with*

$$B_k \leqslant N^{\varepsilon}, \tag{1.12}$$

that is,

$$B_l \leqslant N^{\varepsilon} < B_{l+1}$$

and let m be a natural number satisfying

$$\|mt\| \leqslant \tfrac{1}{2}N^{-\varepsilon}. \tag{1.13}$$

If $l \geqslant 2$, *then m has the form*

$$m = c_l B_{l-1} + c_{l+1} B_l + \cdots. \tag{1.14}$$

In other words, in the representation (1.8),

$$c_1 = c_2 = \cdots = c_{l-1} = 0.$$

Proof. Let $1 \leqslant l' < l$ and set $m' = c_{l'} B_{l'-1} + \cdots$. We may assume $\|m't\| \leqslant \|t\|$. By (1.9) we have

$$\begin{aligned} \|m't\| &= |c_{l'} D_{l'-1} + \cdots| \\ &> |c_{l'} D_{l'-1} + (a_{l'+1} - 1) D_{l'} + a_{l'+3} D_{l'+2} + \cdots \\ &= |(c_{l'} - 1) D_{l'-1} - D_{l'}| \geqslant |D_{l'}|. \end{aligned} \tag{1.15}$$

(I made use of the facts that $\operatorname{sgn} D_k = (-1)^k$ and $-D_k = a_{k+2} D_{k+1} + a_{k+4} D_{k+3} + \cdots$, $k = 0, 1, \ldots$.

From (1.13) it follows that

$$\|m't\| \geqslant |D_{l'}| \geqslant |D_{l-1}|. \tag{1.16}$$

Since (1.7) implies

$$|D_{l-1}| = \frac{1}{B_{l-1}\zeta_l + B_{l-2}} > \frac{1}{B_l + B_{l-1}} > \frac{1}{2B_l} > \frac{1}{2N^{\varepsilon}},$$

Lemma 4 is proved.

In Lemmas 4′, 5, and 6, l ($\geqslant 2$) and N are as in Lemma 4.

LEMMA 4′. *c_l can only be* 0 *or* 1.

Namely, if c_l were $\geqslant 2$, then (1.15) would yield

$$\|mt\| > |D_{l-1}|.$$

LEMMA 5. *Denote, for any interval* $I \subseteq (0, 1)$,

$$A_n(t, I) = \sum_{\substack{\nu \leqslant N \\ \{\nu t\} \in I}} 1.$$

(Here $\{y\}$ denotes the fractional part of y.) If the length of such an I is $|D_l|$, then for $n > |D_l|^{-1}$,

$$A_n(t, I) \leqslant 2\,|D_l|\,n. \tag{1.17}$$

Proof. See Hecke [2].

LEMMA 6. *For $m \leqslant \frac{1}{2}B_{l+1}$, the natural numbers ν for which*

$$\|\nu t\| < \frac{1}{2N^{\varepsilon}}, \tag{1.18}$$

are of the form

$$\nu = kB_l, \qquad k = 1, 2, \ldots . \tag{1.19}$$

Proof. The proof follows immediately from (1.14) with $m = \nu$.

2. THE CONSTRUCTION

I carry out the construction under assumption (3) instead of (3′), where A can be arbitrarily large. The passage to the general case (3′) is obvious.

Denote $e(t) = \exp(2\pi i t)$ and by $g_m(t)$ the $(C, 2)$-mean of the mth partial sum of the geometric series $1 + e(t) + e(2t) + \cdots$, that is,

$$g_m(t) = \binom{m+2}{2}^{-1}\left(\binom{m+2}{2}(1-e(t))^{-1} - (m+1)\,e(t)(1-e(t))^{-2} + e(2t)(1-e((m+1)t)(1-e(t))^{-3}\right). \tag{2.1}$$

It is known that, as $m \to \infty$,

$$\begin{aligned} g_m(t) &= O(m), \qquad \text{uniformly in } t, \\ &= (1-e(t))^{-1} - \frac{2}{m+2}\,e(t)(1-e(t))^{-2} + O\left(\frac{1}{m^2\,\|t\|^3}\right). \end{aligned} \tag{2.2}$$

Further, the real and imaginary parts of

$$(1-e(t))^{-1} - \frac{2}{m+2}\,e(t)(1-e(t)^{-2}) \tag{2.3}$$

are of bounded variation in every closed interval in which $\|t\| > \delta$ provided $m > m_0(\delta)$ ($\delta \in (0, \frac{1}{2})$ arbitrary).

Let A be a large integer and let $0 < k_1 < k_2 < \cdots$ be integers satisfying

$$(A, k_n) = 1 \qquad \text{and} \qquad k_{n+1} > k_n^4. \tag{2.4}$$

Let

$$p_{n1} < p_{n2} < \cdots \tag{2.5}$$

be the finite sequence of primes p satisfying

$$k_n < p < 4k_n \tag{2.6}$$

and

$$p \equiv 1 \pmod{A} \tag{2.7}$$

(for all large n there are such p's by the Prime Number Theorem for arithmetic progressions). For a fixed n, I write p_l instead of p_{nl}. Set, for $\varepsilon > 0$,

$$f_n(t) = k_n^{-3(1+\varepsilon)/2} \sum_{k_n < p_l < 4k_n} e(k^2 lt)\, g_{k_n}(p_l t)\, g_{[k_n/A]}(At) \tag{2.8}$$

and

$$f(t) = \sum_{n=1}^{\infty} f_n(t). \tag{2.9}$$

Obviously $f(t)$ is a lacunary trigonometric series with gaps of length at least A. Since the sum of the squares of the coefficients converges, we have

$$f(t) \in L^2(0, 1). \tag{2.10}$$

Let $0 < \delta < \frac{1}{2}$. I show that the series (2.9) converges uniformly for $\|t\| > \delta$. Let $t \in (\delta, 1-\delta)$ and let k' be the largest k_n for which

$$\|At\| < \delta(2k_n^3)^{-1}. \tag{2.11}$$

(If there is no such largest k_n, we set $k' = 0$. We may suppose that $\|At\| \neq 0$, because otherwise the desired uniform convergence is evident.) Write

$$f(t) = \sum_{n=1}^{\infty} f_n(t) = \sum_{k_n \leqslant k'} + \sum_{k_n > k'} = \sum_1 + \sum_2. \tag{2.12}$$

First, estimate $|\sum_1|$ from above. We have, because of (2.7), for any large n with $k_n \leqslant k'$ and all l,

$$\|p_l t\| = \left\| \left[\frac{p_l}{A}\right] \|At\| + \|t\| \right\| \geqslant \frac{\delta}{2}. \tag{2.13}$$

Because of the bounded variation of the real and imaginary parts of the function (2.3), we have, by partial summation, for $k_n \leqslant k'$

$$|f_n(t)| < \frac{2}{\delta}\frac{k_n}{A} \max_{0\leqslant r\leqslant k_n} \left| \sum_{0\leqslant l\leqslant r} e(k^2 lt) \right|. \tag{2.14}$$

Therefore we only have to estimate $\max_{0\leqslant r\leqslant k_n} |\sum_{0\leqslant l\leqslant r} l(k^2 lt)|$, which, by (2.4) and the inequality $\|k_n^2 t\| > (2A)^{-1}$, does not exceed $4A$. This gives

$$\left|\sum_1\right| = \frac{A}{\delta} O\left(\sum_{k_n\leqslant k'} k_n^{-1/2}\right) \qquad \text{as} \quad n\to\infty. \tag{2.15}$$

For $\sum_2$ we have, for large n,

$$\left|\sum_2\right| \leqslant \sum_{k_n^3 > (\delta/2)|At|^{-1}} \frac{k_n^{\varepsilon/2}}{k_n^{3/2}} \sum_{k_n<p_l<4k_n} |g_{k_n}(p_l t)| \tag{2.16}$$

which is a consequence of (2.4). In order to finish the proof of uniform convergence, I show that

$$\sum_{k_n<p_l<4k_n} |g_{k_n}(p_l t)| = O(k_n). \tag{2.17}$$

If t is rational, $t=a/b$, $(a,b)=1$, then $g_{k_n}(p_l t) < 1/b$ with at most one exception and there is nothing to prove. Therefore we may suppose that t is irrational. We have

$$\sum_{k_n<p_l<4k_n} |g_{k_n}(p_l t)| = \sum_{\|p_l t\|\leqslant 1/40k_n} + \sum_{\|p_l t\|>1/40k_n} = \sum{}' + \sum{}''.$$

Due to Lemma 1, $\sum'$ contains at most one summand; therefore

$$\left|\sum{}'\right| < k_n. \tag{2.18}$$

Let

$$B_m \leqslant k_n < B_{m+1}\ (m\geqslant 0). \tag{2.19}$$

Then

$$\sum{}'' \leqslant \sum_{0\leqslant\mu\leqslant|D_m|^{-1}} \sum_{\|p_l t\|\in I_\mu} |g_{k_n}(p_l t)|,$$

where I_μ is the interval

$$((40k_n)^{-1} + \mu|D_m|, (40k_n)^{-1} + (\mu+I)|D_m|).$$

Each of these intervals contains at most $2D_m k_n$ elements of the sequence $\{p_l t\}$ (because of Lemma 5). Therefore

$$\left|\sum{}''\right| < k_n |D_m| \frac{1}{|D_m|} (1 + \tfrac{1}{2} + \cdots + [|D_m|^{-1}]) \tag{2.20}$$

which proves that we need if $B_{m+1} \leqslant k_n$; if $B_{m+1} > k_n$, then a similar argument involving B_{m-1} instead of B_m leads to our goal. Inequalities (2.18) and (2.20) prove the uniform convergence of (2.9) in $(\delta, 1-\delta)$. Thus $f(t)$ is bounded in each closed subinterval of $(\delta, 1-\delta)$. Finally I show that

$$f \notin L^q(0, 1), \tag{2.21}$$

if $q > 2$ and $\varepsilon < \varepsilon_0(q)$.

By the completeness of $L^q(0, 1)$, it suffices to show that

$$\overline{\lim_{n\to\infty}} \int_0^1 |s_n(x)|^q \, dx = \infty, \tag{2.22}$$

where $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$. Set $\theta_n = \varepsilon' k_n^{-3}$. Then, for $\|x\| < \theta_n$, we have, for a suitable ε',

$$f_r(x) = (1 + O(1))\, k_r^{-3(1+\varepsilon)/2} k_r^2 A^{-1} \sum_{k_r < p_r l < 4k_r} 1,$$

or, after some simple calculations applying the Prime Number Theorem for arithmetic progressions,

$$|s_n(x)| > K \cdot k^{3/2 - 2\varepsilon}, \qquad K \text{ being a constant.}$$

Therefore (2.4) yields

$$\int_0^1 |s_n(x)|^q \, dx > \varepsilon' k_n^{-3} (k_n^{(3/2) - 2\varepsilon})^q K^q$$

which tends to infinity if ε is small enough. This completes the proof.

Acknowledgment

I would like to express my gratitude to Professor Oved Shisha for helping me to write the present paper and eliminate several errors.

REFERENCES

1. P. ERDÖS AND A. RÉNYI, On a problem of A. Zygmund, *in* "Studies in Mathematical Analysis and Related Topics," Essays in Honor of George Pólya, pp. 110–116, Stanford, 1962.
2. E. HECKE, Über Analytische Funktionen und Die Verteilung Der Zahlen Mod Eins, *Hamb. Abh.* **1** (1922), 102–106.
3. A. E. INGHAM, Some trigonometric inequalities with applications to the theory of series, *Math. Z.* **41** (1936), 367–379.
4. S. KNAPOWSKI, On power series with small gaps, *J. London Math. Soc.* **39** (1964), 489–494.
5. A. OSTROWSKI, "Mathematische Miszellen: Notiz zur Theorie der Diophantischen Approximation, *Jahresber. Deutsch. Math.-Verein* **36** (1927), 178–180.
6. O. PERRON, Die Lehre von den Ketterebruchen. I. 3rd ed., Stuttgart, 1954.
7. P. TURÁN, On a certain problem in the theory of power series with gaps, *in* "Studies in Mathematical Analysis and Related Topics," Essays in Honor of George Pólya, pp. 404–409, Stanford, 1962.
8. N. WIENER, A class of gap theorems, *Ann. Pisa* **3** (1934), 367–372.
9. A. ZYGMUND, "Trigonometric Series," Cambridge Univ. Press, London/New York, 1959.